Outline

I. Splines on fans

II. Open & proposed problems

III. Tools old and new

I. Splines on fans

Σ - pure, hereditary, n-dim'l fan in $\mathbb{R}^n$ (rat'n'l polyhedral)

Σ_i° - interior i-dim cones

$R = \mathbb{R}[x_1, \dots, x_n]$, $R \cong \bigoplus_{d \geq 0} R_d$ homog. polys deg d

$\tau \in \Sigma_{n-1}^\circ$, $\alpha_\tau \in R_1$ lin form vanishing on τ

Smoothness distribution $r : \Sigma_{n-1}^\circ \longrightarrow \mathbb{Z}_{\geq -1}$

F diff. to order $r(\tau)+1$ across τ

Splines on (Σ, r):

$$S^r(\Sigma) := \{ F = (F_\sigma)_{\sigma \in \Sigma_n} \in R^{\Sigma_n} : F_{\sigma_1} - F_{\sigma_2} \in \langle \alpha_\tau^{r(\tau)+1} \rangle$$
$$\forall \sigma_1, \sigma_2 \in \Sigma_n \text{ s.t. } \sigma_1 \cap \sigma_2 = \tau \in \Sigma_{n-1}^\circ \}$$

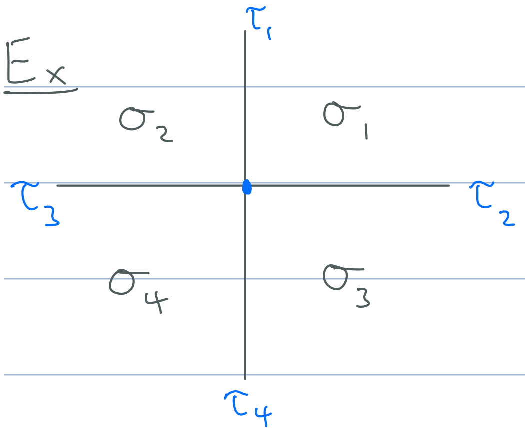
if $r \in \mathbb{Z}_{\geq -1}$, this means $r(\tau) = r \; \forall \tau \in \Sigma_{n-1}^\circ$

$$S^r(\Sigma)_d := \{ F \in S^r(\Sigma) : F_\sigma \in R_d \; \forall \sigma \in \Sigma_n \}$$

f.d. $\mathbb{R}$-v.s.

$$S^r(\Sigma) \cong \bigoplus_{d \geq 0} S^r(\Sigma)_d$$

Ex



$\Sigma \subseteq \mathbb{R}^2$ (complete) $\quad \alpha_{\tau_1} = \alpha_{\tau_4} = x$

$\Sigma_2 = \{\sigma_1, \ldots, \sigma_4\} \quad \alpha_{\tau_2} = \alpha_{\tau_3} = y$

$\Sigma_1 = \Sigma_1^0 = \{\tau_1, \ldots, \tau_4\}$

$\Sigma_0 = \Sigma_0^0 = \{(0,0)\}$

Smthnss dist $r : \Sigma_1^0 \to \mathbb{Z}_{\geq -1}$ by $r(\tau_1) = r(\tau_4) = a$, $r(\tau_2) = r(\tau_3) = b$.

Define $F_1, F_2, F_3, F_4 \in S^r(\Sigma)$ by:

$$F_1 = \begin{array}{c|c} 0 & x^{a+1}y^{b+1} \\ \hline 0 & 0 \end{array} \qquad F_2 = \begin{array}{c|c} y^{b+1} & y^{b+1} \\ \hline 0 & 0 \end{array} \qquad F_3 = \begin{array}{c|c} 0 & x^{b+1} \\ \hline 0 & x^{b+1} \end{array} \qquad F_4 = \begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array}$$

	F_1	F_2	F_3	F_4
deg:	$a+b+2$	$b+1$	$a+1$	0

Matrix form:

$$\begin{array}{c c} & \begin{array}{cccc} F_1 & F_2 & F_3 & F_4 \end{array} \\ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} & \begin{bmatrix} x^{a+1}y^{b+1} & y^{b+1} & x^{a+1} & 1 \\ 0 & y^{b+1} & 0 & 1 \\ 0 & 0 & x^{a+1} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Notice $\det = x^{a+1} y^{b+1} x^{a+1} y^{b+1}$

$$= \prod_{\tau \in \Sigma_1} \alpha_\tau^{r(\tau)+1}$$

[Rose '96]: $S^r(\Sigma)$ is a free R-module,

$$S^r(\Sigma) \cong RF_1 \oplus RF_2 \oplus RF_3 \oplus RF_4$$

$$\cong R(-a-b-2) \oplus R(-b-1) \oplus R(-a-1) \oplus R$$

II. Open Problems

① Compute $\dim S^r(\Sigma)_d$, find basis as $\mathbb{R}$-v.s.

② Determine if $S^r(\Sigma)$ is free, find R-module basis

<u>Appx Thy focus gp:</u> Tackle ①,② when $\Sigma = \Sigma^{\mathcal{A}}$ is fan induced by a <u>central hyperplane arrangement</u> $\mathcal{A} \subseteq \mathbb{R}^n$ $(n=2,3)$. [$\Sigma^{\mathcal{A}}_n$ = <u>chambers</u> of $\mathcal{A}$]

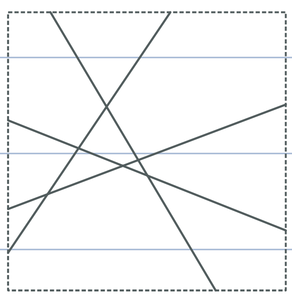
$r : \Sigma^{\mathcal{A}}_{n-1} \to \mathbb{Z}_{\geq -1}$ is <u>constant</u> along hyperplanes

e.g. $\mathcal{A} = \bigcup_{i=1}^{k} H_i$. $\tau_1, \tau_2 \in \Sigma^{\mathcal{A}}_{n-1}$ and $\tau_1, \tau_2 \subseteq H_i$ then $r(\tau_1) = r(\tau_2)$.

(see example!)

<u>Motivation</u>

① "Cross-cut partitions"



Σ = cone over subdivision of region in $\mathbb{R}^2$ by straight lines

[Chui-Wang '83] $\dim S^r_d(\Sigma)$ computed

[Schenck-Stillman '97] $S^r(\Sigma)$ <u>free</u> $\mathbb{R}[x,y,z]$-module

$\mathcal{A} \subseteq \mathbb{R}^3$ hyp. arr, even $S^0(\Sigma^{\mathcal{A}})$ is not completely understood.

② Resolving power ideals. $\mathcal{A} \subseteq \mathbb{R}^3$, $\Sigma = \Sigma^{\mathcal{A}}$, r smthness dist.

$J(0) = \langle \alpha_\tau^{r(\tau)+1} : \tau \in \Sigma_{n-1} \rangle \subseteq \mathbb{R}[x,y,z]$.

$\gamma \in \Sigma^{\mathcal{A}}_1$, $J(\gamma) = \langle \alpha_\tau^{r(\tau)+1} : \tau \in \Sigma_2, \gamma \subseteq \tau \rangle$

[Schenck '97], [D '16]: $S^r(\Sigma^{\mathcal{A}})$ free $\Leftrightarrow$ $\operatorname{syz} J = \sum_{\gamma \in \Sigma^{\mathcal{A}}_1} \operatorname{syz} J(\gamma)$

Connections to literature on

Power ideals	Zonotopal Algebra	Cox-Nagata Rings
[Postnikov-Shapiro '04]	[Holtz-Ron '11]	[Sturmfels-Xu '10]
[Schenck '04]	[Lenz '16]	
[Ardila-Postnikov '12]		

Good candidates here for $S^r(\Sigma^{\mathcal{A}})$ to be free.

III. Tools old & new

① [Billera '88], [Schenck-Stillman '97]

SES of chain complexes $0 \to \mathcal{J}_\bullet \to R_\bullet \to R_\bullet/\mathcal{J}_\bullet \to 0$

Σ complete fan in $\mathbb{R}^3$

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{\tau\in\Sigma_2} J(\tau) \arrow[r, "\partial_2'"] \arrow[d] & \bigoplus\limits_{\gamma\in\Sigma_1} J(\gamma) \arrow[r, "\partial_1'"] \arrow[d] & J(0) \arrow[r] \arrow[d] & 0 \\
\bigoplus\limits_{\sigma\in\Sigma_3} R \arrow[r, "\partial_3"] \arrow[d] & \bigoplus\limits_{\tau\in\Sigma_2} R \arrow[r, "\partial_2"] \arrow[d] & \bigoplus\limits_{\gamma\in\Sigma_1} R \arrow[r, "\partial_1"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
\bigoplus\limits_{\sigma\in\Sigma_3} R \arrow[r, "\partial_3"] & \bigoplus\limits_{\tau\in\Sigma_2} R/J(\tau) \arrow[r, "\partial_2"] & \bigoplus\limits_{\gamma\in\Sigma_1} R/J(\gamma) \arrow[r] & R/J(0) \arrow[r] & 0
\end{tikzcd}$$

$J(\tau) = \langle \alpha_\tau^{r(\tau)+1} \rangle$, $J(\gamma) = \langle \alpha_\tau^{r(\tau)+1} : \gamma \le \tau \rangle$, $J(0) = \langle \alpha_\tau^{r(\tau)+1} : \tau \in \Sigma_2 \rangle$

$H_3(R_\bullet/\mathcal{J}_\bullet) \cong S^r(\Sigma)$, $S^r(\Sigma) \cong R \oplus H_2(\mathcal{J}_\bullet)$, $H_0(\mathcal{J}_\bullet) = 0$

[Schenck '97]: $S^r(\Sigma)$ free $\Leftrightarrow H_1(\mathcal{J}_\bullet) = 0$

(require certain conditions on Σ - for instance, simplicial or $\Sigma = \Sigma^{\mathcal{A}}$)

[D '16]: $H_1(\mathcal{J}_\bullet) \cong \dfrac{\operatorname{syz}(J(0))}{\sum\limits_{\gamma\in\Sigma_1} \operatorname{syz}(J(\gamma))}$

One problem: determine $\dim S^r(\Sigma^{\mathcal{A}})_d$ when $\mathcal{A}$ is generic (no three hyperplanes meet on a line)

Can approach using this.

② [Rose '96] Saito-Rose freeness criterion (saw in example)

Put $t = |\Sigma_n|$.

$F_1, \dots, F_t \in S^r(\Sigma)$ are free R-module basis $\iff$

$$\det \begin{bmatrix} (F_1)_{\sigma_1} & \cdots & (F_t)_{\sigma_1} \\ \vdots & & \vdots \\ (F_1)_{\sigma_t} & \cdots & (F_t)_{\sigma_t} \end{bmatrix} = c \prod_{\tau \in \Sigma^0_{n-1}} \alpha_\tau^{r(\tau)+1}$$

③ [GTV '16] <u>Generalized splines</u>

$G = (V, E)$, $r : E \to \mathbb{Z}_{\geq -1}$, $\ell : E \to R_1$

$$S^r(G, \ell) := \{ F = (F_v)_{v \in V} \in R^V : F_{v_1} - F_{v_2} \in \langle \ell(e)^{r(\tau)+1} \rangle \;\; \forall e \in E,\ e = \{v_1, v_2\} \}$$

- Idea: spline condition separated from geometry

④ (New) Deletion-restriction sequence.

$\Sigma = \Sigma^{\mathcal{A}}$, $H = \mathbb{V}(\alpha_H) \in \mathcal{A}$.

There exists left-exact sequence

$$0 \to S^{r-1_H}(\Sigma) \xrightarrow{\cdot \alpha_H} S^r(\Sigma) \xrightarrow{\pi} S^r(G_\Sigma, \ell|_H)$$

reduce mod α_H

- $(r - 1_H)(\tau) = \begin{cases} r(\tau) & \text{if } \tau \not\subset H \\ r(\tau) - 1 & \text{if } \tau \subset H \end{cases}$

- G_Σ = dual graph of $\Sigma^{\mathcal{A}}$ (edge graph of zonotope)
- $\ell|_H(\tau) = \langle \alpha_\tau^{r(\tau)+1} \rangle + \langle \alpha_H \rangle \subseteq R/\langle \alpha_H \rangle$

Questions (inspired by

1. Surjectivity of π? (almost never)

   local? (also almost never)

2. How to alter $S^r(G_{\Sigma'}, \ell|_H)$ to "fix" 1?

3. Analyze $S^r(G_{\Sigma'}, \ell|_H)$ when $A \subseteq \mathbb{R}^3$ [is $\mathbb{R}[x,y]$-module]

   or its suitable modification

4. Addition-deletion theorems? (motivation comes from $D(A,m)$)

5. Suitable long exact sequences?